Asymptotic dimension of graphs

Louis Esperet (CNRS, Grenoble, France)

(joint work with M. Bonamy, N. Bousquet, C. Groenland, F. Pirot and A. Scott)

Workshop on Graph Theory & Combinatorics in Thuringia
July 31, 2020

A notion of dimension for metric spaces

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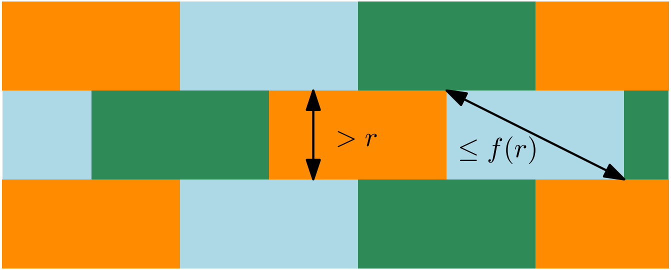

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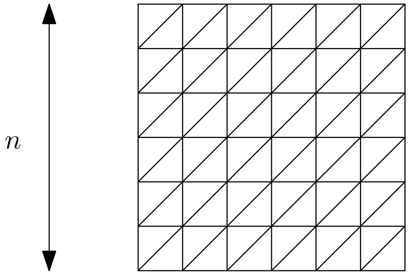

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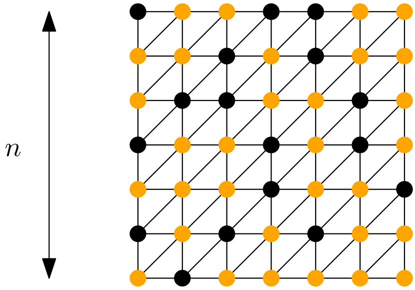

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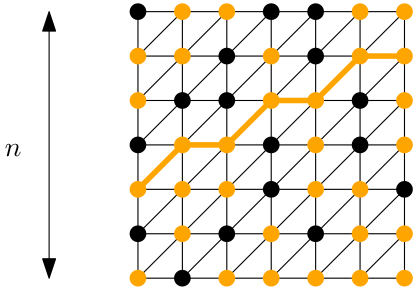

Asymptotic dimension

Gromov (1993). A metric space X has asymptotic dimension at most d if there is a function f such that for every $r > 0$, X can be covered by sets of at most $d+1$ colors, each of diameter at most $f(r)$, and any two sets of the same color are at distance $> r$ apart.

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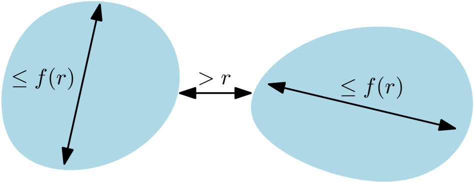

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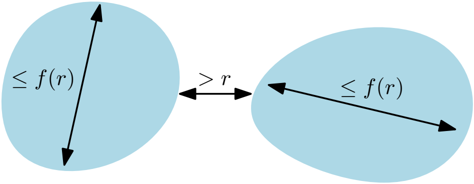

- $\mathbb{R}^d$ and d-dimensional grids should have dimension d
- robust and invariant under local modifications
- a continuous space and a sufficiently fine discretization should have the same dimension
- a clean combinatorial definition

Quasi-isometry

Two metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there is a map $f : X \to Y$ and constants $\epsilon \geq 0$, $\lambda \geq 1$, and $C \geq 0$ such that any element of Y is at distance at most C from some element of $f(X)$, and for every $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) - \epsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \epsilon.$$

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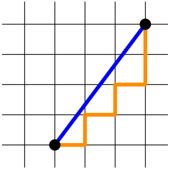
Example. $X =$ the 2-dimensional grid, $Y = \mathbb{R}^2$, f : the grid $\to \mathbb{Z}^2$.

$$\frac{1}{\sqrt{2}} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2).$$



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Observation

Quasi-isometric spaces have the same asymptotic dimension.

Geometric group theory

Given a finitely generated group G and a finite symmetric set of generators S, the Cayley graph of (G, S) has vertex set x and an edge between any element $x \in G$ and any element $xs \in G$ $(s \in S)$.

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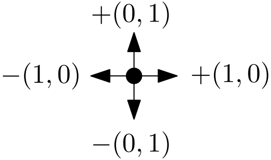

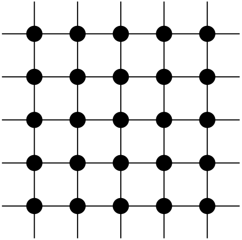

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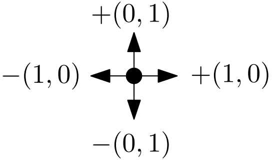

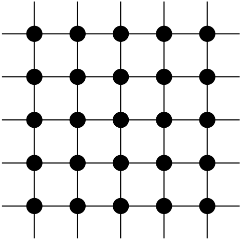

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All Cayley graphs of a finitely generated group G are quasi-isometric, and thus they have the same asymptotic dimension.

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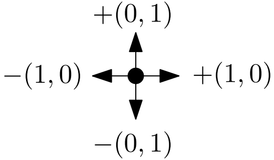

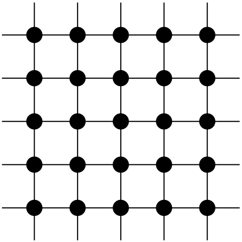

Observation

All Cayley graphs of a finitely generated group G are quasi-isometric, and thus they have the same asymptotic dimension.

So the asymptotic dimension is a group invariant !

A convenient rephrasing

A graph G has asymptotic dimension at most d if there is a function f such that for every $r > 0$, G can be covered by sets of at most $d + 1$ colors, each of diameter at most $f(r)$, and any two sets of the same color are at distance $> r$ apart.

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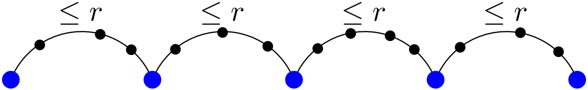

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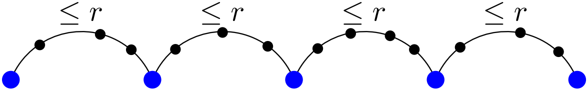

A monochromatic r-component is a maximal set of vertices of the same color, lying in the same component of G^r (the graph obtained from G by adding edges between vertices at distance at most r apart).

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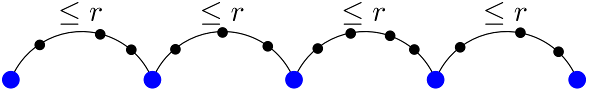

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Equivalent definition

A graph G has asymptotic dimension at most d if there is a function f such that for every $r > 0$, G has a $(d+1)$-coloring, in which each monochromatic r-component has diameter at most $f(r)$ in G.

Basic results

Asymptotic dimension of graph classes

A class of graphs $\mathcal{G}$ has asymptotic dimension at most d if there is a function f such that for every $r > 0$, any graph $G \in \mathcal{G}$ has a $(d+1)$-coloring, in which each monochromatic r-component has diameter at most $f(r)$ in G.

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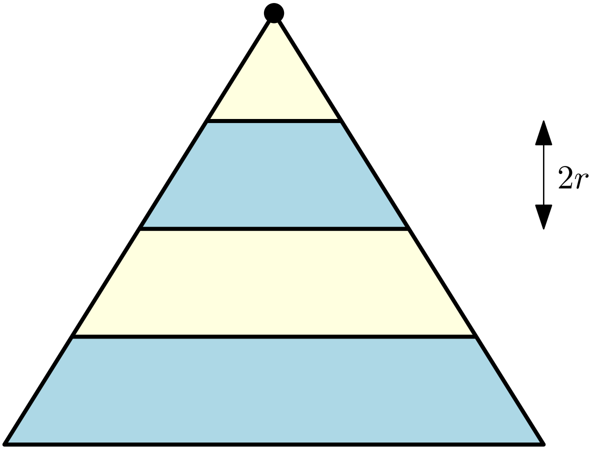
- the class of all finite d-dimensional grids has asymptotic dimension d
- trees have asymptotic dimension 1
- any family of bounded degree expanders has infinite asymptotic dimension (Hume 2017)

Trees



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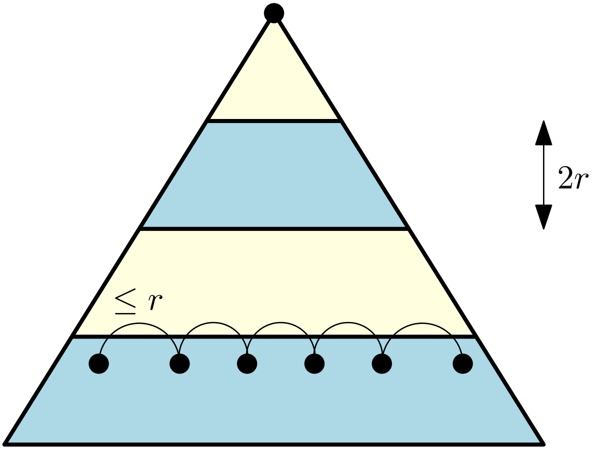

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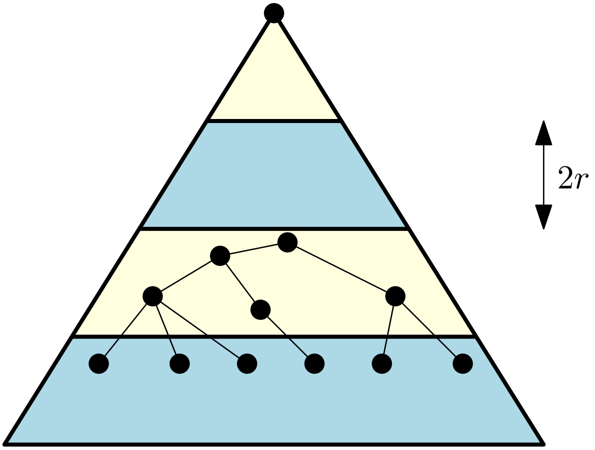
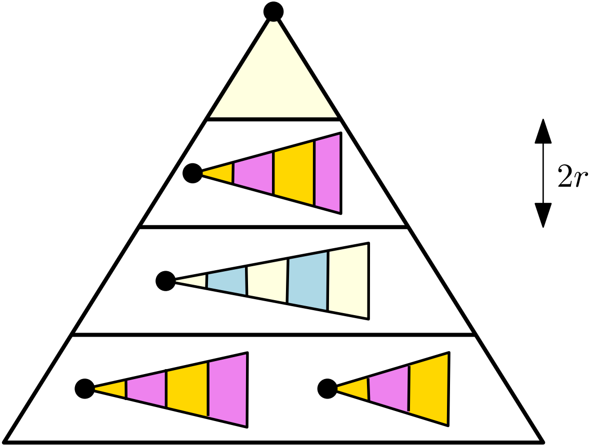


Planar graphs



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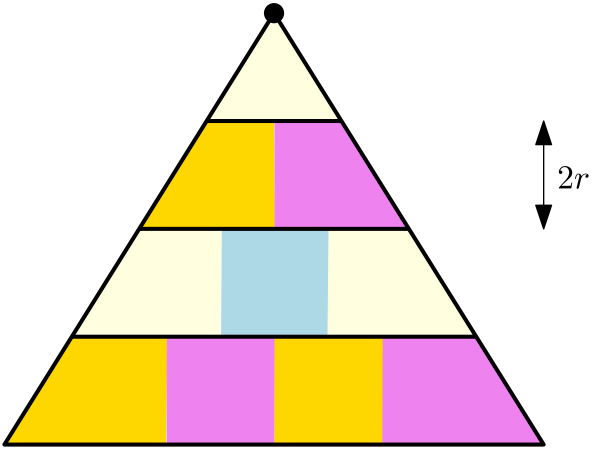

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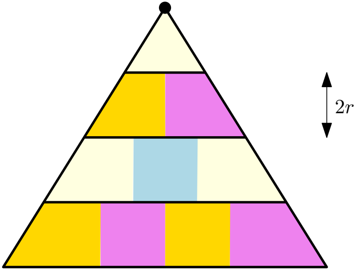

Theorem (Fujiwara, Papasoglou 2020)

Planar graphs have asymptotic dimension at most 3.

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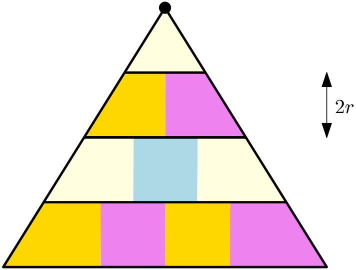

Theorem (Fujiwara, Papasoglou 2020)

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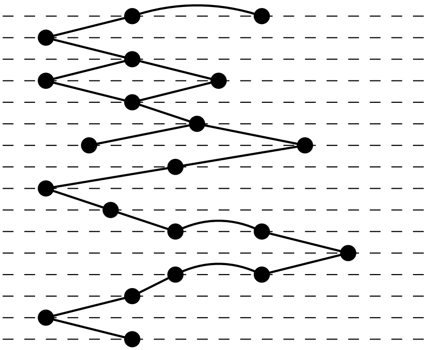
Theorem (Bonamy, Bousquet, E., Groenland, Pirot, Scott 2020)

$\forall k$, $K_{3,k}$-minor free graphs have asymptotic dimension at most 2.

Layerings



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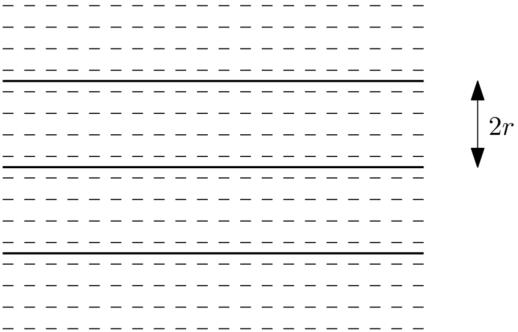

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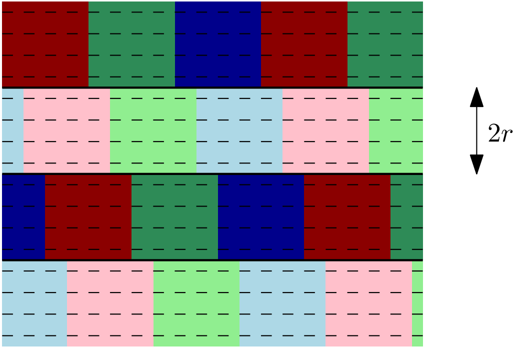

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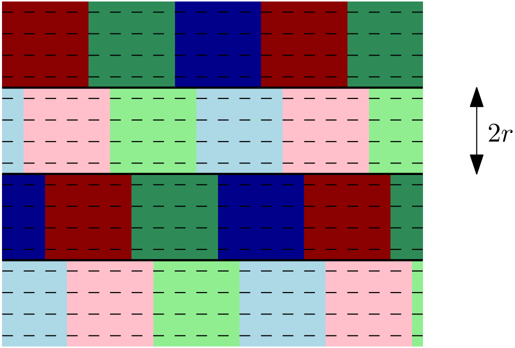

Observation

If for any $r > 0$, the class of graphs induced by $2r$ consecutive layers has asymptotic dimension at most d, then the whole graph has asymptotic dimension at most $2d + 1$.

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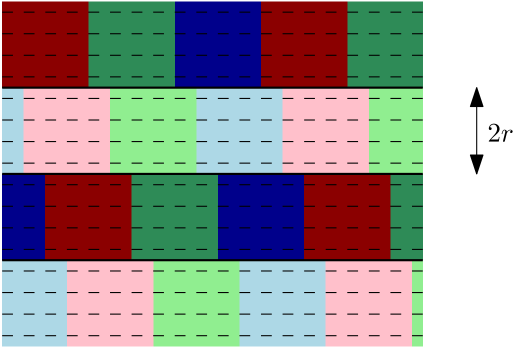

Theorem (Brodskiy, Dydak, Levin, Mitra 2008)

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Treewidth and layered treewidth

Theorem (Bonamy, Bousquet, E., Groenland, Pirot, Scott 2020)

$\forall k, \Delta$, graphs of treewidth k and maximum degree Δ have asymptotic dimension at most 1.

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Graphs of bounded layered treewidth have layerings in which any constant number of consecutive layers induce a graph of bounded treewidth.

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Theorem (Bonamy, Bousquet, E., Groenland, Pirot, Scott 2020)

$\forall k, \Delta$, graphs of layered treewidth k and maximum degree Δ have asymptotic dimension at most 2.

H-minor free graphs of bounded degree

Theorem (Dujmović, E., Morin, Walczak, Wood 2020)

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Question (Ostrovskii Rosenthal 2015)

Is it true than any minor excluded group has asymptotic dimension at most 2?

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Theorem (Liu 2020)

$\forall k$, graphs of treewidth k have asymptotic dimension ≤ 1.

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$\forall H$, H-minor free graphs have asymptotic dimension ≤ 2.

The best known bound for K_t-minor free graphs was asymptotic dimension at most 4^t (Ostrovskii Rosenthal 2015).

Connections with clustered coloring

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If a class has maximum degree Δ and asymptotic dimension at most d, any graph G in the class has a $(d+1)$-coloring such that any monochromatic component has size at most $\text{poly}(\Delta)$.

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Corollary

$\forall H, \Delta$, H-minor free graphs of maximum degree Δ have a 3-coloring such that any monochromatic component has size at most $\text{poly}(\Delta)$.

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$\forall H, \Delta$, H-minor free graphs of maximum degree Δ have a 3-coloring such that any monochromatic component has size at most $\text{poly}(\Delta)$.

This weaker statement was only proved in 2019 for planar graphs (Liu Wood), and in 2020 for H-minor free graphs (DEMWW).

Connections with network decompositions

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If a class has asymptotic dimension at most d, then for any $r > 0$, any graph in the class has a partition into sets of diameter $f(r)$ such that any r-ball intersects at most $d + 1$ sets.

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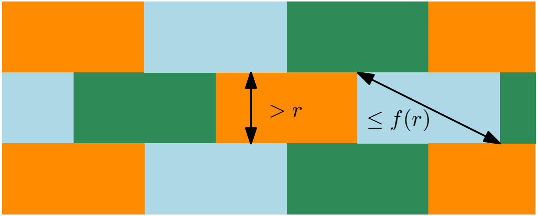

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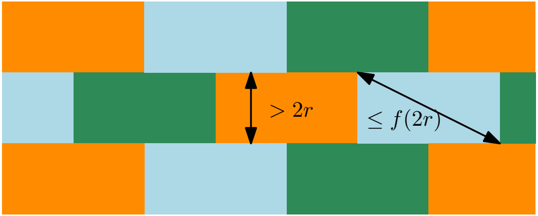

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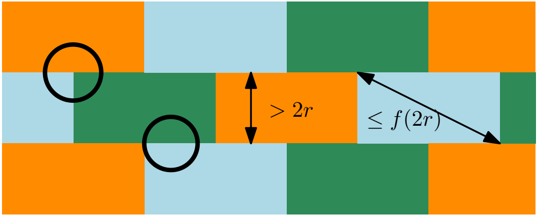

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If a class has asymptotic dimension at most d, then for any $r > 0$, any graph in the class has a partition into sets of diameter $f(r)$ such that any r-ball intersects at most $d+1$ sets.

In many cases we can take $f(r) = O(r)$ (d-dimensional grids, trees, $K_{3,k}$-minor free graphs).

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If $\sigma \cdot d$ is small, then several combinatorial optimisation problems can be approximated efficiently.

Open problem

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We can prove that the answer is positive for classes of polynomial growth (i.e. such that the size of r-balls grows as $O(r^d)$, for some d).