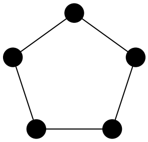
On Vizing's edge-colouring question

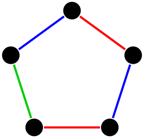
Marthe Bonamy

July 28, 2020

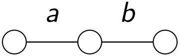
Edge colouring



Edge colouring

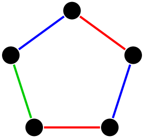


χ': Minimum number of colors to ensure that

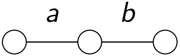


$\Rightarrow a \neq b.$

Edge colouring



χ': Minimum number of colors to ensure that



$\Rightarrow a \neq b$.

Δ: Maximum degree of the graph.

$$\Delta \leq \chi'$$

Vizing's theorem and Kempe equivalence

Theorem (Vizing '64)

For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Vizing's theorem and Kempe equivalence

Theorem (Vizing '64)

For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

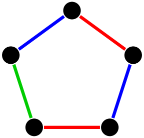
Proof through "Kempe changes".

Vizing's theorem and Kempe equivalence

Theorem (Vizing '64)

For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof through "Kempe changes".

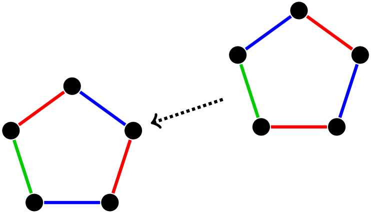


Vizing's theorem and Kempe equivalence

Theorem (Vizing '64)

For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof through "Kempe changes".

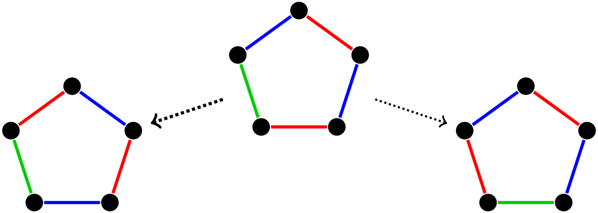


Vizing's theorem and Kempe equivalence

Theorem (Vizing '64)

For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof through "Kempe changes".

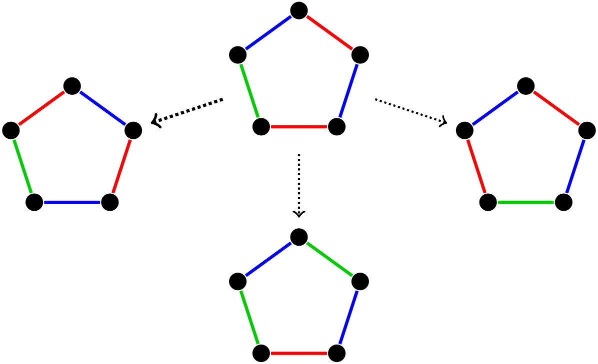


Vizing's theorem and Kempe equivalence

Theorem (Vizing '64)

For any graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof through "Kempe changes".



Vizing's theorem, revisited

Theorem (Vizing '64)

For any graph G, for any proper edge colouring α of G, there is a proper $(\Delta(G) + 1)$-edge colouring β of G such that α and β are Kempe-equivalent.

Vizing's theorem, revisited

Theorem (Vizing '64)

For any graph G, for any proper edge colouring α of G, there is a proper $(\Delta(G)+1)$-edge colouring β of G such that α and β are Kempe-equivalent.

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Vizing's theorem, revisited

Theorem (Vizing '64)

For any graph G, for any proper edge colouring α of G, there is a proper $(\Delta(G)+1)$-edge colouring β of G such that α and β are Kempe-equivalent.

Theorem (Misra Gries '92 (Inspired from the proof))

For any simple graph $G = (V, E)$, a $(\Delta + 1)$-edge-coloring can be found in $\mathcal{O}(|V| \times |E|)$.

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Vizing's theorem, revisited

Theorem (Vizing '64)

For any graph G, for any proper edge colouring α of G, there is a proper $(\Delta(G)+1)$-edge colouring β of G such that α and β are Kempe-equivalent.

Theorem (Misra Gries '92 (Inspired from the proof))

For any simple graph $G = (V, E)$, a $(\Delta + 1)$-edge-coloring can be found in $\mathcal{O}(|V| \times |E|)$.

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Theorem (Holyer '81)

It is NP-complete to compute χ'.

Mohar's conjecture

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Mohar's conjecture

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Only interesting for $\chi'(G) = \Delta(G)$.

Mohar's conjecture

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Only interesting for $\chi'(G) = \Delta(G)$.

Conjecture (Mohar '06)

For any graph G, for any two $(\Delta(G) + 2)$-edge colourings α and β of G, they are Kempe-equivalent.

Mohar's conjecture

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Only interesting for $\chi'(G) = \Delta(G)$.

Conjecture (Mohar '06)

For any graph G, for any two $(\Delta(G)+2)$-edge colourings α and β of G, they are Kempe-equivalent.

True if $\chi'(G) = \Delta(G)$.

Mohar's conjecture

Conjecture (Vizing '65)

For any graph G, for any proper edge colouring α of G, there is a proper $\chi'(G)$-edge colouring β of G such that α and β are Kempe-equivalent.

Only interesting for $\chi'(G) = \Delta(G)$.

Conjecture (Mohar '06)

For any graph G, for any two $(\Delta(G) + 2)$-edge colourings α and β of G, they are Kempe-equivalent.

True if $\chi'(G) = \Delta(G)$.

(Vizing's conjecture) $\Rightarrow$ (Mohar's conjecture): induction on $\Delta(G)$.

Small Delta

Theorem (McDonald, Mohar, Scheide '10)

Vizing's conjecture is true for $\Delta = 3$.

Small Delta

Theorem (McDonald, Mohar, Scheide '10)

Vizing's conjecture is true for $\Delta = 3$.

Theorem (Asratian, Casselgren '16)

Vizing's conjecture is true for $\Delta = 4$.

Small Delta

Theorem (McDonald, Mohar, Scheide '10)

Vizing's conjecture is true for $\Delta = 3$.

Theorem (Asratian, Casselgren '16)

Vizing's conjecture is true for $\Delta = 4$.

Theorem (B., Defrain, Klimošová, Lagoutte, Narboni '20)

Vizing's conjecture is true for triangle-free graphs.

Small Delta

Theorem (McDonald, Mohar, Scheide '10)

Vizing's conjecture is true for $\Delta = 3$.

Theorem (Asratian, Casselgren '16)

Vizing's conjecture is true for $\Delta = 4$.

Theorem (B., Defrain, Klimošová, Lagoutte, Narboni '20)

Vizing's conjecture is true for triangle-free graphs.

Theorem (B., Defrain, Klimošová, Lagoutte, Narboni '20)

For any triangle-free graph, all $(\chi' + 1)$-edge-colourings are Kempe-equivalent.

General structure

- By induction on $\chi'(G)$.

General structure

- By induction on $\chi'(G)$.
- It suffices to consider $\chi'(G)$-regular graphs.

General structure

- By induction on $\chi'(G)$.
- It suffices to consider $\chi'(G)$-regular graphs.
- Consider a target $\chi'(G)$-edge-colouring α, and M one of its color classes (M is a perfect matching).

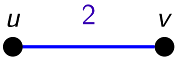
General structure

- By induction on $\chi'(G)$.
- It suffices to consider $\chi'(G)$-regular graphs.
- Consider a target $\chi'(G)$-edge-colouring α, and M one of its color classes (M is a perfect matching).
- Goal: make M monochromatic (say with colour 1).

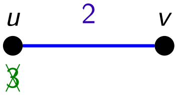
General structure

- By induction on $\chi'(G)$.
- It suffices to consider $\chi'(G)$-regular graphs.
- Consider a target $\chi'(G)$-edge-colouring α, and M one of its color classes (M is a perfect matching).
- Goal: make M monochromatic (say with colour 1).
- Good ($\in M$, coloured 1), bad ($\in M$, not coloured 1), ugly ($\notin M$, coloured 1) edges.

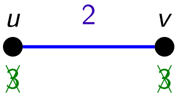
Fan-like tools



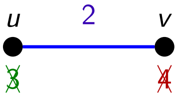
Fan-like tools



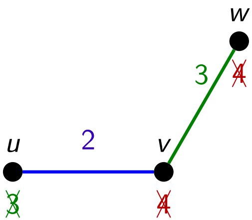
Fan-like tools



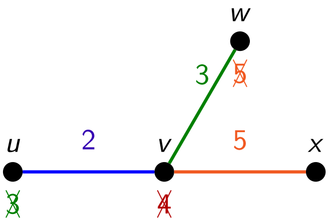
Fan-like tools



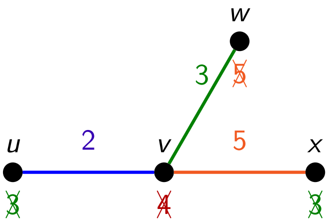
Fan-like tools



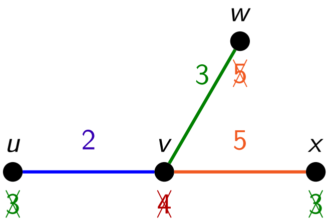
Fan-like tools



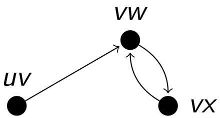
Fan-like tools



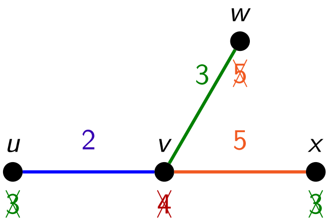
Fan-like tools



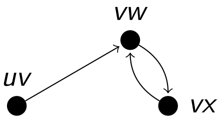
$\overrightarrow{D_v}$: $vy \rightarrow vz$ if vz is coloured with the colour missing at y.



Fan-like tools



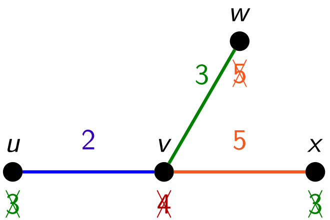
$\overrightarrow{D_v}$: $vy \to vz$ if vz is coloured with the colour missing at y.



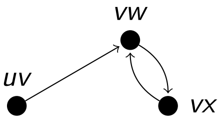
X_u: sequence of vertices of $\overrightarrow{D_v}$ reached from uv.

- Path: 😎

Fan-like tools



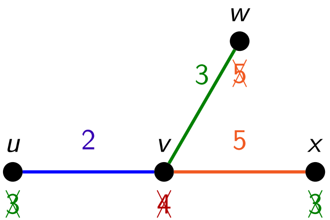
$\overrightarrow{D_v}$: $vy \rightarrow vz$ if vz is coloured with the colour missing at y.



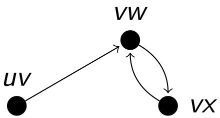
X_u: sequence of vertices of $\overrightarrow{D_v}$ reached from uv.

- Path: 😎
- Cycle: 😕

Fan-like tools



$\overrightarrow{D_v}$: $vy \to vz$ if vz is coloured with the colour missing at y.

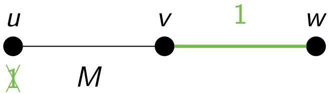


X_u: sequence of vertices of $\overrightarrow{D_v}$ reached from uv.

- Path: 😎
- Cycle: 😕
- Comet: 😎 (sort of)

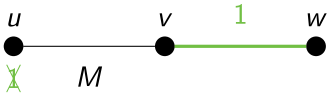
Back to the general picture

We can argue the existence of:



Back to the general picture

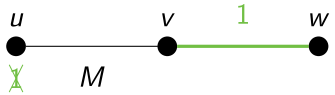
We can argue the existence of:



Then X_v in $\overrightarrow{D_w}$ is a cycle.

Back to the general picture

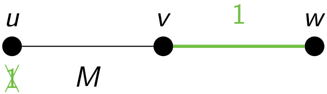
We can argue the existence of:



Then X_v in $\overrightarrow{D_w}$ is a cycle. X_w in $\overrightarrow{D_v}$ is also a cycle.

Back to the general picture

We can argue the existence of:



Then X_v in $\overrightarrow{D_w}$ is a cycle. X_w in $\overrightarrow{D_v}$ is also a cycle.

Two cycles $\Rightarrow$ 😎 (unless there is a triangle vwx with $\cancel{1}$ at x).

Conclusion

Conclusion

Danke schön!