The extremal function for sparse minors

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Erfurt (sort of) 28th July 2020

Long ago ...

Everything is a graph - no loops or multiple edges

H is a *minor* of G written $H \prec G$ if H can be obtained from G by a sequence of deletions and edge-contractions

Mader (60s) asked: how many edges in G guarantee $K_t \prec G$?

Mader: $\exists\, c(t)$ such that $e(G) \geq c(t)|G|$ implies $K_t \prec G$

$c(t) \leq 2^{t-3}$ (Mader 67), $\quad c(t) \leq 8t\log_2 t$ (Mader 68)

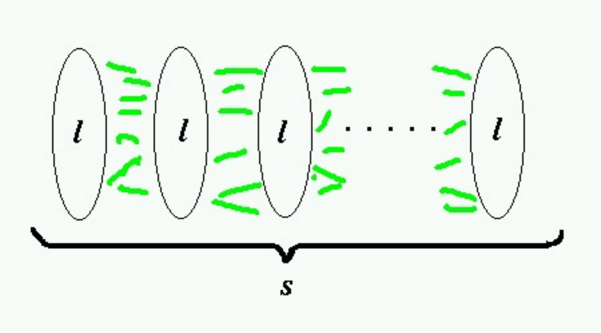
$G = K_{t-2} + \overline{K}_{n-t+2}$ shows $c(t) \geq t-2$

$c(3) = 1 \;\; c(4) = 2 \;\; c(5) = 3 \;\; c(6) = 4 \;\; c(7) = 5 \;\; \ldots$ (Mader 68)

$c(t) \geq\ c\, t\sqrt{\log t}$ (Bollobás+Catlin+Erdős 80)

Why $\sqrt{\log}$?

Let $G = G(n,p)$ be random. Is $K_s \prec G$? Let $\ell = n/s$.



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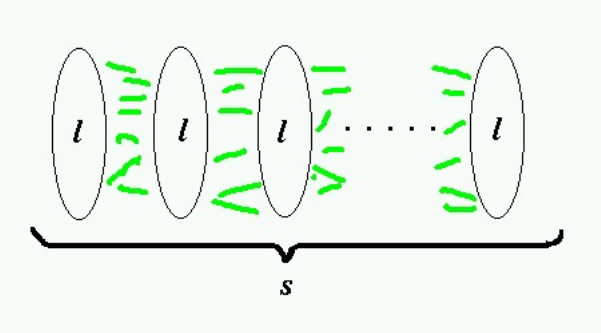

$\Pr\{\text{two blobs have no edge between}\} = (1-p)^{\ell^2}$

If we put $\ell = \sqrt{(1-\epsilon)\log s / \log(1-p)}$ this is $s^{-1+\epsilon}$ so

$$\begin{aligned}
\Pr\{K_s \prec G\} &\le \text{number of blobbings} \times \Pr\{\text{blobbing is ok}\} \\
&\le s^n \times (1 - s^{-1+\epsilon})^{\binom{s}{2}} \\
&\le \exp\{s\ell \log s - s^{-1+\epsilon}\binom{s}{2}\} = o(1)
\end{aligned}$$

The value of $c(t)$

$c(t) \geq \ 0.319\, t \sqrt{\log t}$ (Bollobás+Catlin+Erdős 80)

where $0.319\ldots = \max_{p>0} \frac{p/2}{\sqrt{\log 1/(1-p)}}$ (at $p = 0.715\ldots$)

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$c(t) \; = \; \Theta(t \sqrt{\log t})$ (Kostochka 82, T 84)

$c(t) \; = \; (0.319 + o(1))\, t \sqrt{\log t}$ (T 01)

Extremal graphs are (more or less) disjoint unions of random-like graphs of the optimal size+density (Myers 02)

Incomplete minors

OK it's known that $c(t) = (0.319 + o(1))\, t \sqrt{\log t}$

Given H, define $c(H)$ by $\quad e(G) \geq c(H)|G|$ implies $H \prec G$

Let H have t verts and ave degree d. Clearly $c(H) \leq c(t)$.

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Define $\gamma(H) \;=\; \min_w \frac{1}{t} \sum_{u \in H} w(u)$, where $w : V(H) \to \mathbf{R}^+$ and

$$\sum_{uv \in E(H)} e^{-w(u)w(v)} \leq t$$

Note $\gamma(H) \leq \sqrt{\log d}$

If $\quad d \geq t^{\epsilon} \quad$ then $c(t) \;=\; (0.319 + o(1))\, t\, \gamma(H)$ (Myers+T 05)

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If $\quad d \geq t^\epsilon \quad$ then $\gamma(H) \approx \sqrt{\log d}$ for almost all H

If $\quad d \geq t^\epsilon \quad$ then $\gamma(H) \approx \sqrt{\log d}$ all regular H

$\gamma(K_{\beta t, (1-\beta)t}) \sim 2\sqrt{\beta(1-\beta)\log t}$

Sparse minors

If $\quad d \geq t^{\epsilon} \quad$ then $c(H) \leq (0.319 + o(1))\, t \sqrt{\log d} \quad$ (Myers+T 05)

What if d smaller, say $d = \log t$, eg if $H =$ hypercube?

Sparse minors

If $\quad d \geq t^{\epsilon} \quad$ then $c(H) \leq (0.319 + o(1))\, t \sqrt{\log d} \qquad$ (Myers+T 05)

What if d smaller, say $d = \log t$, eg if $H =$ hypercube?

$$\Pr\{H \prec G(n,p)\} \leq \text{number of blobbings} \times \Pr\{\text{blobbing is ok}\}$$

d small $\Longrightarrow \Pr\{\text{blobbing is ok}\}$ is large $\Longrightarrow$ first term dominates

In fact
$d \leq \log t \Longrightarrow G(t, 1/2)$ contains a *spanning* $H \quad$ (Alon+Füredi 92)

Sparse examples

$c(K_{2,t}) = \frac{t+1}{2}$ — Myers 03 large t; Chudnovsky+Reed+Seymour 11, all t

$c(K_{s,t}) = (\frac{1}{2} + o(1))t$ — Kühn+Osthus 05, large t

$c(K_{s,t}) = \frac{t+3s}{2} + O(\sqrt{s})$ — Kostochka+Prince 07, large t

$c(K_{s,t}) \leq \frac{t+6s\log s}{2}$
true for $s \leq ct/\log t$
false for $s > Ct\log t$ — Kostochka+Prince 10

$c(\text{hypercube}) = O(t)$ (Hendrey+Norin+Wood 19+)

Get on with it

If $\quad d \geq t^{\epsilon} \quad$ then $c(H) \leq (0.319 + o(1))\, t\sqrt{\log d}$ (Myers+T 05)

For all H, $c(H) \leq 3.895\, t\sqrt{\log d}$ (Reed+Wood 15)

Get on with it

If $\quad d \geq t^{\epsilon} \quad$ then $c(H) \leq (0.319 + o(1))\, t\sqrt{\log d} \qquad$ (Myers+T 05)

For all H, $c(H) \leq 3.895\, t\sqrt{\log d} \qquad$ (Reed+Wood 15)

Theorem (Wales+T 20+)

Given $\epsilon > 0$ there exists d_0 such that, for all $d \geq d_0$:
all graphs H of order t and average degree $d > d_0$ satisfy

$$c(H) \leq (0.319 + \epsilon)\, t\sqrt{\log d}$$

Theorem (Norin+Reed+T+Wood 20)

Given $\epsilon > 0$ there exists d_0 such that, for all $d \geq d_0$:
for all $t \geq d$, almost all graphs H of order t and average degree d satisfy

$$c(H) \geq (0.319 - \epsilon)\, t\sqrt{\log d}$$

The lower bound

G is a *blowup* of a *tiny* random graph   (c.f. Fox 11)

Take $G_0 = G(d, 0.715\ldots)$

Form G by blowing up vertices of G_0
so that G has average degree $0.319t\sqrt{\log d}$

Show $H \not\prec G$ for almost all H $\langle$*insert maths here*$\rangle$

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Is this a contradiction in maths?
Ie G is extremal so it should be pseudo-random

The upper bound

Lemma (Wales+T)

Given $\epsilon > 0$ there exists d_0 such that, for all $d \geq d_0$:
if G is a graph of density at least $p + \epsilon$, with $\kappa(G) \geq \epsilon|G|$ and $|G| \geq t\sqrt{\log_{1/(1-p)} d}$, then $G \succ H$ for all H order t and ave deg d.

The upper bound

Lemma (Wales+T)

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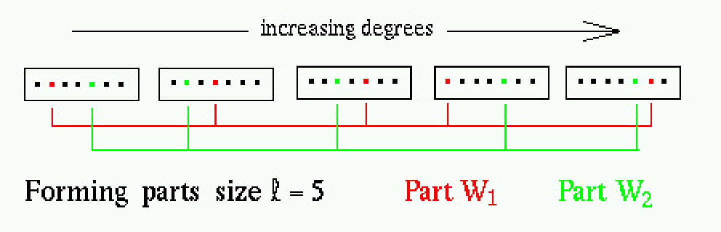
Proof.

a) "Degree random" partition G: t parts W_i, $|W_i| = \ell = |G|/t$



b) Randomly map $V(H)$ to $\{W_1, \ldots, W_t\}$. □

Thanks for your attention